TREATISE I:
THE PLANE MIRROR & MULTIPLE IMAGES FROM DOUBLE-MIRROR REFLECTIONS

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1. The Plane Mirror

1.1. **Parallax.**

1.1.1. *Location of an object by parallax.* When viewing objects that lie in a line, the relative position of the objects can be ascertained through the use of parallax.

While viewing the object, shift your view either to the left or the right. The closer item(s) will move contrary to the change in perspective while the farther item(s) will move in the same direction as the change in perspective.

For example, assume there exists a pencil (†) at arms length from you, and a pen (‡) about 15cm closer. When seen straight on there would appear to be a single object. Upon moving your head to the right you see the pencil to the right of the pen(‡ †), and when moving your head to the left you see the pencil to the left of the pen († ‡). For example, in Figure 1, position A is looking from the left, position B is looking straight on, and position C is looking from the right.

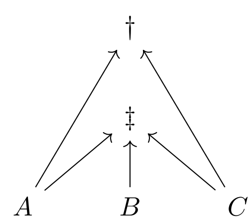

Figure 1. The parallax between a pen (‡) and pencil (†).

Further, if there is no parallax between two objects (i.e. if there is no apparent motion between the objects) they must be at the same position (as seen in Figure 2).

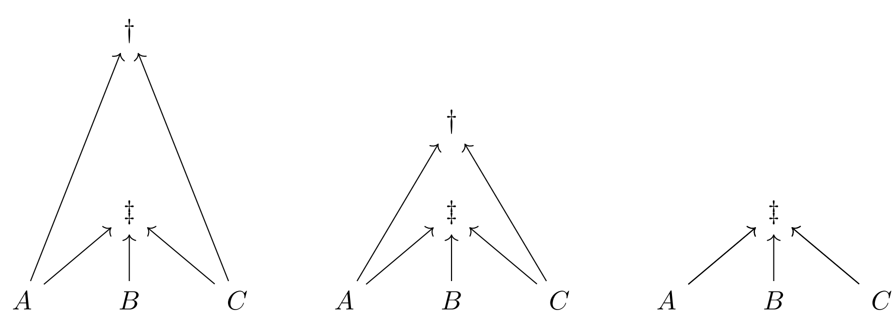

Figure 2. There is no parallax if and only if the two objects are at the same position.

1.1.2. *Location of an image by parallax.* Parallax can also be used to locate an image as seen in a plane mirror. For example, assume there exists a pin (p_1) that is standing about 10cm in front of a plane mirror; upon moving your head left and right (from a position where the pin and its image lie in a straight line) the image will appear to move the same direction as your head. From our understanding of parallax, this means that the image exists behind the real pin. Further, by using a second pin (p_2), you can find the location of the p_1's image by moving p_2 around until there exists no parallax between p_1's image and p_2. After doing so, you will see that the distance between p_1 and p_2 is equal to twice the distance from p_1 to the plane mirror, in other words, the plane mirror is equidistant from both p_1 and p_2.

1.2. **Ray Tracing.**

1.2.1. *Location of an object by ray tracing.* It is possible to find the location of an object by constructing a ray diagram that shows the path along which the light travels to our eyes. Experimentally, this can be done by sticking a pin (P_O) that will act as our object vertically into a piece of paper on a sheet of cardboard. From here, insert two additional pins at a time into the cardboard along your line of sight so that pairs of pins fall in line with P_O, do this three times from widely different directions. After lining up the pins draw in the lines passing through the pins and you will see that they converge at P_O, our object.

1.2.2. *Location of an image by ray tracing.* The same technique of following rays that we used to find the location of an object can be used to determine the location of our object's image. Continuing with the same setup as before, stand a planar mirror on the cardboard, and mark its location. Look into the mirror and locate the image of the pin (P_I), from here place pairs of pins in line with the image as you did with the object in the previous section. After you have at least three pairs of pins in line with P_I, remove the mirror and trace the lines the pins created. The three lines should intersect at a point and equal distance from the mirror as our object, P_O, was from the mirror, and along a line perpendicular to the mirror that passes through P_O. This is the location of P_I.

1.3. **Relationship of Angles.** By measuring the angles from the previous section, we can conclude that the the angle of entry is equal to the angle of exit in the mirror, as can be seen in Figure 3 which models this setup.

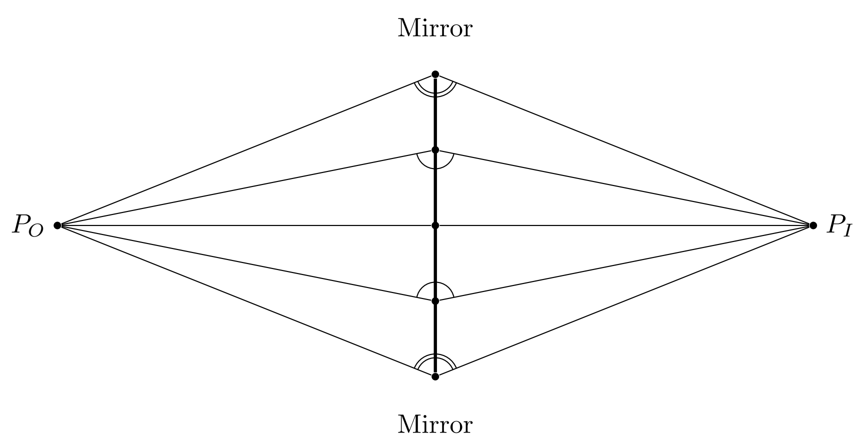

Figure 3. Single arc angles are about 79° and double arc angles are about 69°.

1.4. **Putting it All Together.** It is possible to locate the image of an object placed in front of a planar mirror in two ways, though the use of parallax (finding the position that exhibits no parallax with the image) and though the the use of ray tracing. (Both approaches should result in the same image location.) The image generated by the mirror is an erect virtual image of the same size as the object. The reason we call this type of image *virtual* is that nothing would appear on a screen placed at the image point, in contrast to a *real* image. Additionally, in the case of planar mirrors the image will appear to be behind the mirror, and the distance between the image and the mirror is equal to the distance between the mirror and the object. Also mirrors obey the *law of reflection*, that the angle of incidence and angle of reflection are equal, and this holds for any optic (planar *or* curved).

1.5. **Ray Diagrams.**

1.5.1. *Theory of the ray diagram.* In addition to being able to experimentally locate an image, it is also possible to locate an image though the use of theoretical ray diagrams. For example, consider an erect object and mirror such as in Figure 4, the image will be behind the mirror a distance equal to that between the mirror and the object.

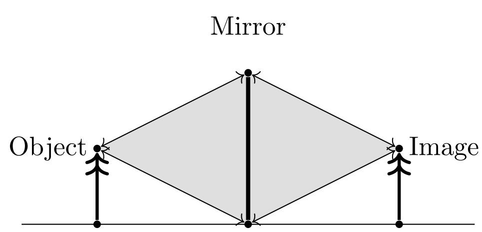

Figure 4. The image and object are equidistant from the mirror, and are of equal size.

1.5.2. *Checking our your new clothes.* This same concept can reveal that in order to see your entire body in a mirror you will need a mirror at least half as tall as you are, as can be seen in Figure 5. The distance between the person and the mirror will not affect the necessary size, it will however make the image appear farther away. For every x meters away from the mirror you stand, your reflection will appear to be $2x$ meters away.

1.6. **Some puzzles.**

1.6.1. *Viewing Angle.* As usual, the image will appear and equal distance 'behind' the mirror as the object is in front, except that does not lie directly behind the physical mirror, but slightly off to the side. The region from which this image can be seen in the mirror is contained within the lines that intersect at the image, and pass through the edges of the mirror, as in Figure 6.

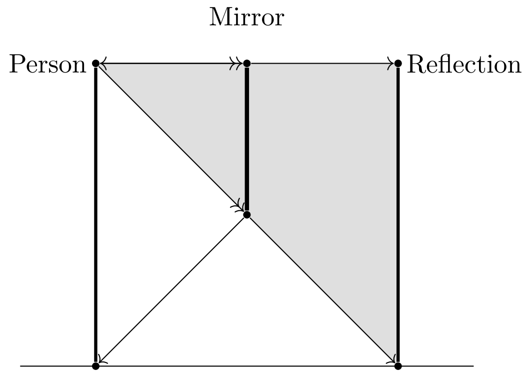

FIGURE 5. To "check out your new clothes," you'll need a mirror half as tall as yourself.

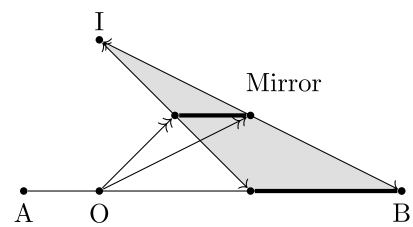

FIGURE 6. The image can be seen from anywhere within the bold portion of $\overline{AB}$.

1.6.2. *Using Mirrors.* It is possible to both draw a perpendicular to a straight line and draw a tangent to a curve using a mirror without simply using the mirror as a straightedge. In the case of drawing a perpendicular to a straight line on a piece of paper, we would position the mirror on the line and rotate it until the line on the paper and the line viewed in the mirror come into alignment, at which point the plane of our mirror follows the perpendicular to the line. In the case of drawing a tangent to a curve, we would position the mirror on the curve at the point we wanted a tangent, and then rotate the mirror until the line going into the mirror and the line seen in the mirror appeared straight at the point of intersection, this gives us the perpendicular to the tangent, and we can then use the previous method to find the tangent from the tangent's perpendicular.

2. Multiple Images from Double-Mirror Reflections

2.1. **Setting up the apparatus.** In the following experiments we will be using two small mirrors, taped along one edge such that they can be stood at various angles in a $\vee$-shape. Place a sheet of polar coordinate graph paper on a piece of cardboard, and place the mirrors such that the vertex of the $\vee$ is at the origin. We will be utilizing a 9V battery as our three-dimensional object with a clearly identifiable left and right sides, and a clearly identifiable front and back. We will also be utilizing a pin, and this will be our precision object.

2.2. **Perverted and Normal Images.**

2.2.1. *Differences Between Image and Object.* After placing the three-dimensional object on the 90° line in front of the mirrors while they are opened along the $0° - 180°$ line, you will observe that there is a single perverted image. The image being perverted means that left and right are reversed, causing text to appear backwards.

2.2.2. *Multitudes of Images.* When folding the mirrors together until the angle becomes as small as possible, the number of images rises at an increasing rate. This is because the number of images and the angle between the mirrors are inversely related. While observing the increasing number of images, it can also be noted that 'odd' images are perverted (those that appear 1^{st}, 3^{rd}, 5^{th}, etc.) while 'even' images are normal (those that appear 2^{nd}, 4^{th}, 6^{th}, etc.). Further, all of the images appear to be an equal distance from the point at which the two mirrors meet, and (when considering complete images) they are distributed evenly around the center.

2.3. **Quantitative Position Measurements – Odd Number of Images.** We will now begin to use our precision object. Beginning at 180°, we bring the mirrors together symmetrically until three complete images appear. At this point the angular measure between the mirrors is 90°. While continuing to bring the mirrors together it can be noted that the number of complete images increases more raplidly as the angle gets smaller. Subsequent complete images appear at 60° and 45°, resulting in 5 and 7 complete images respectively.

2.4. **Quantitative Position Measurements – Even Number of Images.** Following the same method as in the previous section, we find at 120°, 72°, and 52°, there occur 2, 4, and 6 images respectively.

2.5. **Analyzing Your Data.** From these observations we conclude that there is a relationship between the number of images produced and the angular measure between the mirrors. This relationship can be seen in Figure 7.

$$A = \frac{360°}{I+1}$$

Figure 7. Where A is the angular distance between the mirrors required to create I images ($I \in \mathbb{Z}$).

2.6. **Ray Diagrams.**

2.6.1. *Two and Three Images.* When placed along the bisector of the mirrors' ∨, the two image ray diagram creates an equilateral triangle and the three image diagram creates a square. This is because there are $I + 1$ points of interest (due to the original object), and as they lie on the bisector the points create a regular polygon.

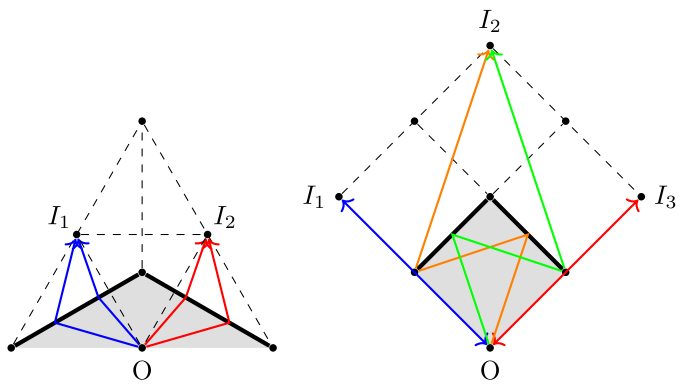

FIGURE 8. The ray diagrams for the two and three image cases.

2.6.2. *Four Images & "Image of an Image".* We begin to notice, beginning with the third and fourth image reflections (as in FIGURE 8 and FIGURE 9), that images after the first on either side are created through the reflection of other images. In essence, images can be used as objects when they fall in 'front' of the other mirrors until they fall into the 'dead zone' that is 'behind' both mirrors.

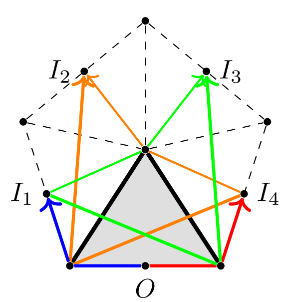

FIGURE 9. The ray diagram for the four image case.

2.6.3. *Number of Reflections & Predicting Perversion.* The number of reflections needed to create an image is equal to how many images back the image in question is. For example, if there are a total of 5 images, the middle image would be the third image back (counting from either the left or right) and would thus require three reflections. Further, the number of reflections can be used to determine if an image is perverted or not as images generated by an odd number of reflections are perverted, while images generated by an even number of reflections are normal.

2.7. **The Secret Shortcut.** In order to predict the locations of images, it is possible to use a *secret shortcut* where one can utilize images as objects in order to locate other images. This will work regardless of the position of the object or angle of the mirror (within physical limits). All images have been found when both chains of images are in the 'dead zone' behind both mirrors where they can not be reflected again.